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GENERALIZED OSBORN LOOPS OF ORDER $4N$

A.O. ISERE, J.O. ADÉNÍRAN, T.G. JAIYÉQLÁ

ABSTRACT. The smallest non-associative Osborn loop is of order 16. Attempts in the past to construct higher orders have been very difficult. In this work, we develop a new method of constructing examples of generalized Osborn loops of order $4n$. Two of such examples are presented. They are shown to be non-associative Osborn loops. These are further classified up to isomorphism to establish their existence as distinct Osborn loops of order $4n$.

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Keywords: Osborn loops, classification, isomorphism.

1. INTRODUCTION

By a loop $G(\cdot)$ we shall mean a non-empty set G together with a binary operation (\cdot) such that the following properties hold: (i) given $a, b \in G$ the equations $a \cdot x = b, y \cdot a = b$ have unique solutions x, y respectively, in G ; (ii) $G(\cdot)$ possesses an identity element, i.e. there exists $e \in G$ such that $e \cdot x = x \cdot e = x$ for all $x \in G$ [27]. An overview of loop theory can be found in Jaiyéqlá [13].

A loop is called an Osborn loop [26, 3] if it obeys any of the following:

$$(x^\lambda \setminus y) \cdot zx = x(yz \cdot x) \quad (1)$$

or

$$x(yz \cdot x) = (x \cdot yE_x) \cdot zx \quad \forall x, y, z \in G \quad (2)$$

where $E_x = R_x R_{x\rho} = (L_x L_x^\lambda)^{-1} = R_x L_x R_x^{-1} L_x^{-1}$

Among the class of Bol-Moufang type of loops is the Bol loop. A loop L is called a Bol loop if and only if:

$$(xy \cdot z)y = x(yz \cdot y) \quad \text{for all } x, y, z \in L \quad (3)$$

Strictly speaking, (3) defines a right Bol loop. A left Bol-loop (L, \cdot) is defined as:

$$x(y \cdot xz) = (x \cdot yx)z \quad \text{for all } x, y, z \in L \quad (4)$$

A Bol loop refers to a left or right Bol loop. The loop that satisfies both (3) and (4) is called a Moufang loop. Therefore, the necessary and sufficient condition for a loop to be a Moufang loop is that the loop is both a left Bol loop and right Bol loop. The smallest order for which a non-associative finite Bol loop exist is 8. There are exactly six Bol loops of order 8 that are not associative These loops were classified by Burn [4]. Solarin and Sharma [30] determined and classified all Bol loops of order 12 that are not associative.

Every Moufang loop is a Bol loop. Therefore, a good knowledge of the classes of Moufang loops becomes indispensable in the classification of Bol loops. Chein [5, 6] found that Moufang loops of orders p , p^2 , p^3 or pq (where p and q are primes) must be groups and by using combinatorial type methods discovered 13 Moufang loops of order ≤ 31 . Purtill [28] has shown that Moufang loops of orders pqr and p^2q where p , q and r are distinct odd primes with $p \leq q \leq r$ are groups. See- [1] for detail.

It is to be noted that a Moufang loop is a variety of Osborn loops. Some of the earliest examples of infinite Osborn loops were constructed by Huthnance [8] in 1968. Other examples of Osborn loops can be found in Isere et al[9, 10]. Thus, examples of Osborn loops are still very few. These examples are presented in Huthnance [8], Isere et. al. [9, 10]. Some recent studies on this class of loops are by Adeniran and Isere [2], Jaiyéolá [15, 17, 18], Jaiyéolá and Adéníran [22, 19, 20], Jaiyéolá et. al. [23]. The application of some identities in universal Osborn loops to cryptography were reported in Jaiyéolá [14, 16], Jaiyéolá and Adéníran [21].

The generalized Osborn loops of order $4n$ are a "k-construction" Osborn loops of order $4n$, where k is any integer. Given an integer k , we have a distinct Osborn loop of order $4n$ constructed in this way. Thus, a generalized Osborn loop gives k number of Osborn loops of order $4n$ constructed this way. In constructing Osborn loops in this way (as it is done in this work), 'a' and 'b' are non-negative variable integers while 'c' is a fixed integer. However, the combination $b+c$ is peculiar to Osborn loops constructed in this way. These loops are found to be non-universal Osborn loops except when $k = 1$. This work is aimed at developing a new method of constructing non-associative, generalized Osborn loops of order $4n$. Two of such examples are presented in the next section. They are shown to be non-associative Osborn loops. Furthermore, the constructed examples are classified upto isomorphism.

2. MAIN RESULTS

2.1. Generalized Osborn Loops

Example 1. Let $I(\cdot) = C_{2n} \times C_2$, $I = \{(x^\alpha, y^\beta), 0 \leq \alpha \leq 2n - 1, 0 \leq \beta \leq 1\}$ such that the binary operation $I(\cdot)$ is defined as follows:

$$(x^a, e) \cdot (x^b, y^\beta) = (x^{a+b}, y^\beta) \quad (5)$$

$$(x^a, y^\alpha) \cdot (x^b, e) = (x^{a+b}, y^\alpha) \quad (6)$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 0 \pmod{2} \quad (7)$$

$$= (x^{a+kb}, y^{\alpha+\beta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 1 \pmod{2} \quad (8)$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+kb}, y^{\alpha+k\beta}) \text{ if } a \equiv 1 \pmod{2}, b \equiv 1 \pmod{2} \quad (9)$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 0 \pmod{2} \quad (10)$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+kb+c}, y^{\alpha+\delta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 1 \pmod{2} \quad (11)$$

$$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha) = (x^{b+c+ka}, y^{\beta+\gamma+k\alpha}) \text{ if } a \equiv 1 \pmod{2}, b \equiv 0 \pmod{2} \quad (12)$$

$$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha) = (x^{c+ka+kb}, y^{\alpha+k\beta+\gamma}) \text{ if } a \equiv 1 \pmod{2}, b \equiv 1 \pmod{2} \quad (13)$$

where k is any integer. Then $I(\cdot)$ is an Osborn loop of order $4n$, where $n = 2, 3, 4, 6, 9, 12$ and 18 .

Proof. We first show that $I(\cdot)$ satisfies Osborn identity below:

$$(X^\lambda \setminus Y) \cdot ZX = X(YZ \cdot X)$$

Now, we begin:

1. Let $X = (x^a, e)$; $Y = (x^b, e)$; $Z = (x^c, e)$, then by direct computations, we have

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, e)^\lambda \setminus (x^b, e)] \cdot [(x^c, e)(x^a, e)].$$

Let $[(x^a, e)^\lambda \setminus (x^b, e)] = (x^d, e)$, then $(x^b, e) = (x^a, e)^\lambda (x^d, e) = (x^{d-a}, e)$ implies $b = d - a$; $d = a + b$.

$$\therefore [(x^a, e)^\lambda \setminus (x^b, e)] = (x^{a+b}, e) \Rightarrow (x^{a+b}, e) \cdot (x^{a+c}, e) = (x^{2a+b+c}, e).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, e)[(x^b, e)(x^c, e) \cdot (x^a, e)] = (x^a, e)[(x^{b+c}, e) \cdot (x^a, e)] = \\ &= (x^a, e)(x^{a+b+c}, e) = (x^{2a+b+c}, e). \end{aligned}$$

2. Let $X = (x^a, e)$; $Y = (x^b, e)$; $Z = (x^c, y^\gamma)$.

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, e)^\lambda \setminus (x^b, e)] \cdot [(x^c, y^\gamma)(x^a, e)].$$

Let $[(x^a, e)^\lambda \setminus (x^b, e)] = (x^d, e)$, then $(x^b, e) = (x^a, e)^\lambda (x^d, e) = (x^{d-a}, e)$ implies $b = d - a$; $d = a + b$.

$$\therefore [(x^a, e)^\lambda \setminus (x^b, e)] = (x^{a+b}, e) \Rightarrow (x^{a+b}, e) \cdot (x^{a+c}, y^\gamma) = (x^{2a+b+c}, y^\gamma).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, e)[(x^b, e)(x^c, y^\gamma) \cdot (x^a, e)] = (x^a, e)[(x^{b+c}, y^\gamma) \cdot (x^a, e)] = (x^a, e)(x^{a+b+c}, y^\gamma) = (x^{2a+b+c}, y^\gamma).$$

3. Let $X = (x^a, e)$; $Y = (x^b, y^\beta)$; $Z = (x^c, e)$.

First Case: when b is even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, e)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, e)(x^a, e)].$$

Let $[(x^a, e)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, e)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore [(x^a, e)^\lambda \setminus (x^b, y^\beta)] = (x^{a+b}, y^\beta) \Rightarrow (x^{a+b}, y^\beta) \cdot (x^{a+c}, e) = (x^{2a+b+c}, y^\beta).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, e)[(x^b, y^\beta)(x^c, e) \cdot (x^a, e)] = (x^a, e)[(x^{b+c}, y^\beta) \cdot (x^a, e)] = (x^a, e)(x^{a+b+c}, y^\beta) = (x^{2a+b+c}, y^\beta).$$

The results of the remaining cases are established in the same way as above.

Second Case: when b is odd $(X^\lambda \setminus Y) \cdot ZX = (x^{2a+kb+c}, y^\beta)$ and $X(YZ \cdot X) = (x^{2a+kb+c}, y^\beta)$.

Let $X = (x^a, y^\alpha)$; $Y = (x^b, e)$; $Z = (x^c, e)$.

First Case: when a is even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, e)] = (x^d, y^\delta)$, the $(x^b, e) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^{\delta-\alpha})$ implies $b = d - a$ and $0 = \delta - \alpha$; implies that $d = a + b$, $\delta = \alpha$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] = (x^{a+b}, y^\alpha) \Rightarrow (x^{a+b}, y^\alpha) \cdot (x^{a+c}, y^\alpha) = (x^{2a+b+c}, e).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, e)(x^c, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{a+b+c}, y^\alpha) = (x^{2a+b+c}, e).$$

Second Case: when a is odd

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, e)] = (x^d, y^\delta)$, then $(x^b, e) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{-(a+ka)}, y^{-\alpha})(x^d, y^\delta) = (x^{d-(ka)}, y^{\delta-\alpha})$ implies $b = d - (ka)$ and $0 = \delta - \alpha$; implies that $d = ka + b, \delta = \alpha$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] = (x^{ka+b}, y^\alpha)(x^{ka+b}, y^\alpha) \cdot (x^{a+c}, y^\alpha) = (x^{a+ka+b+c}, e).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, e)(x^c, e) \cdot (x^a, y^\alpha)] = \\ &= (x^a, y^\alpha)[(x^{b+c}, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{ka+b+c}, y^\alpha) = \\ &= (x^{a+ka+b+c}, e). \end{aligned}$$

Let $X = (x^a, e); Y = (x^b, y^\beta); Z = (x^c, y^\gamma)$.

First Case: when b is even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, e)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, e)].$$

Let $[(x^a, e)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, e)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore [(x^a, e)^\lambda \setminus (x^b, y^\beta)] = (x^{a+b}, y^\beta) \Rightarrow (x^{a+b}, y^\beta) \cdot (x^{a+c}, y^\gamma) = (x^{2a+b+c}, y^{\beta+\gamma}).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, e)[(x^b, y^\beta)(x^c, y^\gamma) \cdot (x^a, e)] = \\ &= (x^a, e)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, e)] = (x^a, e)(x^{a+b+c}, y^{\beta+\gamma}) = \\ &= (x^{2a+b+c}, y^{\beta+\gamma}). \end{aligned}$$

Second Case: when b is odd

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, e)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, e)].$$

Let $[(x^a, e)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, e)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore [(x^a, e)^\lambda \setminus (x^b, y^\beta)] = (x^{a+b}, y^\beta)(x^{a+b}, y^\beta) \cdot (x^{a+c}, y^\gamma) = (x^{2a+kb+c}, y^{\beta+\gamma}).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, e)[(x^b, y^\beta)(x^c, y^\gamma) \cdot (x^a, e)] = \\ (x^a, e)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, e)] &= (x^a, e)(x^{a+kb+c}, y^{\beta+\gamma}) = \\ &= (x^{2a+kb+c}, y^{\beta+\gamma}). \end{aligned}$$

Let $X = (x^a, y^\alpha); Y = (x^b, e); Z = (x^c, y^\gamma)$.

First Case: when a is even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, e)] = (x^d, y^\delta)$, then $(x^b, e) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^{\delta-\alpha})$ implies $b = d - a$ and $0 = \delta - \alpha$; implies that $d = a + b, \delta = \alpha$.

$$\begin{aligned} \therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] &= (x^{a+b}, y^\alpha) \Rightarrow (x^{a+b}, y^\alpha) \cdot (x^{a+c}, y^{\alpha+\gamma}) = \\ &= (x^{2a+b+c}, y^\gamma). \end{aligned}$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, e)(x^c, y^\gamma) \cdot (x^a, y^\alpha)] = \\ (x^a, y^\alpha)[(x^{b+c}, y^\gamma) \cdot (x^a, y^\alpha)] &= (x^a, y^\alpha)(x^{a+b+c}, y^{\alpha+\gamma}) = (x^{2a+b+c}, y^\gamma). \end{aligned}$$

Second Case: when a is odd

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, e)] = (x^d, y^\delta)$, the $(x^b, e) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{-(ka)}, y^{-\alpha})(x^d, y^\delta) = (x^{d-(ka)}, y^{\delta-\alpha})$ implies $b = d - (ka)$ and $0 = \delta - \alpha$; implies that $d = ka + b, \delta = \alpha$.

$$\begin{aligned} \therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, e)] &= (x^{ka+b}, y^\alpha) \Rightarrow (x^{ka+b}, y^\alpha) \cdot (x^{a+c}, y^{\alpha+\gamma}) = \\ &= (x^{a+ka+b+c}, y^\gamma). \end{aligned}$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, e)(x^c, y^\gamma) \cdot (x^a, y^\alpha)] = \\ (x^a, y^\alpha)[(x^{b+c}, e) \cdot (x^a, y^\alpha)] &= (x^a, y^\alpha)(x^{ka+b+c}, y^{\alpha+\gamma}) = (x^{a+ka+b+c}, y^\gamma). \end{aligned}$$

Let $X = (x^a, y^\alpha); Y = (x^b, y^\beta); Z = (x^c, e)$.

First Case: when a and b are even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{a+b}, y^{\alpha+\beta}) \Rightarrow (x^{a+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^\alpha) = (x^{2a+b+c}, y^\beta)$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = \\ (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] &= (x^a, y^\alpha)(x^{a+b+c}, y^{\alpha+\beta}) = (x^{2a+b+c}, y^\beta). \end{aligned}$$

Second Case: when a is odd and b is even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-ka}, y^\delta)$ implies $b = d - (a + ka)$; $d = ka + b$ and $\delta = \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{ka+b}, y^{\alpha+\beta}) \Rightarrow (x^{ka+b}, y^\beta) \cdot (x^{a+c}, e) = (x^{a+ka+b+c}, y^\beta).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = \\ (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] &= (x^a, y^\alpha)(x^{ka+b+c}, y^\beta) = (x^{a+ka+b+c}, y^\beta). \end{aligned}$$

Third Case: when a is even and b is odd

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{a+b}, y^{\alpha+\beta}) \Rightarrow (x^{a+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^\alpha) = (x^{2a+kb+c}, y^\beta).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = \\ (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] &= (x^a, y^\alpha)(x^{a+kb+c}, y^{\alpha+\beta}) = (x^{2a+kb+c}, y^\beta). \end{aligned}$$

Fourth Case: when a and b are odd

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-(ka)}, y^{\delta-\alpha})$ implies $b = d - (ka)$; $d = ka + b$ and $\delta = \alpha + \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{ka+b}, y^{\alpha+\beta}) \Rightarrow (x^{ka+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^\alpha) = (x^{a+ka+kb+c}, y^\beta).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = \\ &= (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{ka+kb+c}, y^{\alpha+k\beta}) = \\ &= (x^{a+ka+kb+c}, y^\beta). \end{aligned}$$

Let $X = (x^a, y^\alpha)$; $Y = (x^b, y^\beta)$; $Z = (x^c, y^\gamma)$.

First Case: when a and b are even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{a+b}, y^{\alpha+\beta}) \Rightarrow (x^{a+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^{\alpha+\gamma}) = (x^{2a+b+c}, y^{\beta+\gamma}).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, y^\gamma) \cdot (x^a, y^\alpha)] = \\ &= (x^a, y^\alpha)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{a+b+c}, y^{\alpha+\beta}) = \\ &= (x^{2a+b+c}, y^{\beta+\gamma}). \end{aligned}$$

Second Case: when a is odd and b is even

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-(a+ka)}, y^\delta)$ implies $b = d - (a + ka)$; $d = a + ka + b$ and $\delta = \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{ka+b}, y^{\alpha+\beta}) \Rightarrow (x^{ka+b}, y^\beta) \cdot (x^{a+c}, y^{\alpha+\gamma}) = (x^{a+ka+b+c}, y^\beta).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, y^\gamma) \cdot (x^a, y^\alpha)] = \\ &= (x^a, y^\alpha)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha)] \\ &= (x^a, y^\alpha)(x^{ka+b+c}, y^{\beta+\gamma}) = (x^{a+ka+b+c}, y^{\beta+\gamma}). \end{aligned}$$

Third Case: when a is even and b is odd

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{a+b}, y^{\alpha+\beta}) \Rightarrow (x^{a+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^{\alpha+\gamma}) = (x^{2a+kb+c}, y^{\beta+\gamma}).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, y^\gamma) \cdot (x^a, y^\alpha)] = \\ &= (x^a, y^\alpha)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{a+kb+c}, y^{\alpha+k\beta+\gamma}) = \\ &= (x^{2a+kb+c}, y^{\beta+\gamma}). \end{aligned}$$

Fourth Case: when a and b is odd

$$(X^\lambda \setminus Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-ka}, y^{\delta-\alpha})$ implies $b = d - (ka)$; $d = ka + b$ and $\delta = \alpha + \beta$.

$$\therefore [(x^a, y^\alpha)^\lambda \setminus (x^b, y^\beta)] = (x^{ka+b}, y^{\alpha+\beta}) \Rightarrow (x^{ka+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^{\alpha+\gamma}) = (x^{a+ka+kb+c}, y^{\beta+\gamma}).$$

$$\begin{aligned} \text{Next, } X(YZ \cdot X) &= (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = \\ &= (x^a, y^\alpha)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{ka+kb+c}, y^{\alpha+k\beta+\gamma}) \\ &= (x^{a+ka+kb+c}, y^{\beta+\gamma}). \end{aligned}$$

Since $(X^\lambda \setminus Y) \cdot ZX = X(YZ \cdot X)$ in all the 36 cases considered i.e. whenever $37 \equiv 1 \pmod{2n}$, then $n = 2, 3, 4, 6, 9, 12, 18$. Also, (e, e) is the two sided identity.

Moreover, if $X = (x^a, e)$, then $X^{-1} = (x^{-a}, e)$. If $X = (x^a, y^a)$ then

$$X^{-1} = (x^{-a}, y^{-a}) \text{ if } a = \text{even and } X^{-1} = (x^{-(ka)}, y^{-a}) \text{ if } a = \text{odd.}$$

Therefore, the inverses are defined. Also for non-associativity, let $X = (x^a, y^\alpha)$; $Y = (x^b, y^\beta)$; $Z = (x^c, y^\gamma)$ where a is an even integer and b an odd integer, then

$$(XY)Z = [(x^a, y^\alpha)(x^b, y^\beta)](x^c, y^\gamma) = (x^{a+kb}, y^{\alpha+\beta})(x^c, y^\gamma) = (x^{a+kb+c}, y^{\alpha+\beta+\gamma}) \text{ and}$$

$$X(YZ) = (x^a, y^\alpha)[(x^b, y^\beta)(x^c, y^\gamma)] = (x^a, y^\alpha)(x^{b+c}, y^{\beta+\gamma}) = (x^{a+b+c}, y^{\alpha+\beta+\gamma})$$

Therefore, $(XY)Z \neq X(YZ)$.

Remark 1. Thus, the Example 1 is non-associative except when $n = 2$ which gives the group $C_4 \times C_2$. Generally, whenever $k = 1$ the examples become associative Osborn loops. Hence, they are non-associative Osborn loops of order $4n$, $n = 4, 6, 9, 12, 18$.

Example 2. Let $I(\cdot) = C_{2n} \times C_2$, $I = \{(x^\alpha, y^\beta), 0 \leq \alpha \leq 2n - 1, 0 \leq \beta \leq 1\}$ such that the binary operation (\cdot) is defined as follows:

$$(x^a, e) \cdot (x^b, y^\beta) = (x^{a+b}, y^\beta) \quad (14)$$

$$(x^a, y^\alpha) \cdot (x^b, e) = (x^{a+b}, y^\alpha) \quad (15)$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 0 \pmod{2} \quad (16)$$

$$= (x^{a+b+kb}, y^{\alpha+\beta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 1 \pmod{2} \quad (17)$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+b+kb}, y^{\alpha+k\beta}) \text{ if } a \equiv 1 \pmod{2}, b \equiv 1 \pmod{2} \quad (18)$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 0 \pmod{2} \quad (19)$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+kb+c}, y^{\alpha+\delta}) \text{ if } a \equiv 0 \pmod{2}, b \equiv 1 \pmod{2} \quad (20)$$

$$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha) = (x^{a+ka+b+kb+c}, y^{\alpha+k\beta+\gamma}) \text{ if } a \equiv 1 \pmod{2}, b \equiv 1 \pmod{2} \quad (21)$$

Then, $I(\cdot)$ is an Osborn loop of order $4n$, where $n = 4, 6, 9, 12$ and 18 , and k any integer.

Proof. The proof is similar to that in Example 1 above.

2.2. Classification up to Isomorphism

Two loops shall be considered non-isomorphic if they contain different number of elements of the same order. Whenever, two loops contain the same number of elements we shall go further to consider the order of elements in their nuclei. If these coincide in both cases, we shall consider commutative patterns of both loops.

Theorem 1. The Osborn loops in Examples 1 and 2 are non-isomorphic.

Proof. (i) Example 1

$$\begin{aligned} (x^a, y^\alpha) \cdot (x^a, y^\alpha) &= ((x^{2a}, y^{2\alpha})) \\ (x^a, y^\alpha) \cdot (x^a, y^\alpha) &= ((x^{2a}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 0 \pmod{2} \\ &= (x^{a+ka}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 1 \pmod{2} \end{aligned} \quad (22)$$

Obviously, the only possible solution to the equation (22) are $a = 0$ and $a = n$ i.e. $(x^{2n}, e) = (e, e)$ and $(e, y^\alpha) = (e, e)$. Therefore, Example 1 has 3 elements

of order 2 whenever k is a positive odd number and 2 elements of order 2 whenever k is positive even number and $k = -2$ and above; $n + 3$ elements of order 2 whenever $k = -1$ and 2 elements whenever k is any negative number except $k = -1$.

(ii) Example 2

$$\begin{aligned} (x^a, y^\alpha) \cdot (x^a, y^\alpha) &= ((x^{2a}, y^{2\alpha})) \\ (x^a, y^\alpha) \cdot (x^a, y^\alpha) &= ((x^{2a}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 0 \pmod{2}) \\ &= (x^{2a+ka}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 1 \pmod{2} \end{aligned} \quad (23)$$

Obviously, the only possible solution to the equation (23) are $a = 0$ and $a = n$ i.e. $(x^{2n}, e) = (e, e)$ and $(e, y^\alpha) = (e, e)$. When k is both positive and odd even integers, we have (x^n, e) , (x^n, y^α) and (e, y^α) as elements of order 2. When $k = 0$, it has 3 elements of order 2. And when $k < 0$, it has 2 elements of order 2. Therefore, Examples 1 and 2 are non-isomorphic since they contain different number elements of the same order.

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