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REPRESENTATIONS OF FINITE OSBORN LOOPS

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ABSTRACT. It is shown that an Osborn loop of order n has $n/2$ generators. Given the determining permutations, the representation Π is generated by $R(2) \circ R(2+i) = R(3+i) \forall i = 1, 3, 5, ..., n-3$. The representation of Osborn loops of order 16 is presented and it is used as an example to verify the results. It is also shown that the order of every element of the representation Π divides the order of the loop, hence, Osborn loops of order 16 are langrangelike.

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1. INTRODUCTION

1.1 Groupoids, Groups, Quasigroups And Loops

A loop L is a quasigroup with a neutral element. All groups are loops but all loops are not groups. Those that are groups are called associative loops. Thus, loop theory is a generalization of group theory by introducing non-associativity into the set. However, we wish to formally define a loop.

Definition 1.1[10]: A loop is a set G with binary operation (denoted here simply by juxtaposition) such that

- for each a in G, the left multiplication map $L_a : G \to G, x \to ax$ is bijective,
- for each a in G, the right multiplication map $R_a : G \to G, x \to xa$ is bijective; and
- G has a two-sided identity in G.

The order of G is its cardinality $|G|$.

Definition 1.2[20]: A loop $(G, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

(i): $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$ for all $x, y \in G$,

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(ii): $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$ for all $x, y \in G$ and
(iii): $x\backslash x = y/y$ or $e \cdot x = x$ and $x \cdot e = x$ for all $x, y \in G$.

It must be stipulated that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x \cdot (y\backslash z)$ respectively.

1.2 Latin squares

A latin square is an n by n array on n symbols having the property that each symbol appears exactly once in each column and row. That is, no element is repeated along each column and row[1]. A latin square must be of finite order, having the same number of elements along the row and column. A set with a binary operation whose multiplication table is a Latin square is called a quasigroup. If there is a two sided identity, the quasigroup is a loop.

Latin squares L_1 and L_2 are in the same isotopy class if some permutation of the rows, columns and symbols of L_1 yields L_2. In order for a set of the n^2 triples to correspond to a Latin square, each pair of columns of the orthogonal array must contain each ordered pair of symbols exactly once. The number of isomorphism classes of quasigroups and loops are obtained in [18]

Definition 1.3[20, 21]: A set Π of permutations on a set G is the representation of a loop $(G, \cdot)$ if and only if

(i): $I \in \Pi$ (identity mapping),
(ii): Π is transitive on G(i.e for all $x, y \in G$, there exists a unique $\pi \in \Pi$ such that $x\pi = y$),
(iii): if $\alpha, \beta \in \Pi$ and $\alpha\beta^{-1}$ fixes one element of G, then $\alpha = \beta$.

The left and right representation of a loop G is denoted by

$$\Pi_\lambda(G, \cdot) = \Pi_\lambda(G) \qquad \text{and} \qquad \Pi_\rho(G, \cdot) = \Pi_\rho(G) \text{ respectively.}$$

Definition 1.4: Let G be a loop. The set $\Pi =\{R(a) : a \in G\}$ is called the right regular representation of G or briefly the representation of G.

1.3 Osborn Loops

A loop $I(\cdot)$ is called an Osborn loop [19] if it obeys the identity:

$$(x^\lambda\backslash y) \cdot zx = x(yz \cdot x) \tag{1}$$

for all $x, y, z \in I$. Here, x^λ is the left inverse of x, and $a\backslash b$ is the left division operation. The term Osborn loops first appeared in a work of Huthnance Jr [9] in 1968, on generalized Moufang loops.

However, the equation (1) above is according to Basarab [5] in 1979. For detail see Kinyon [14] and Jaiyeola [11]. Moreover, the most popularly known varieties of Osborn loops are CC-loops, Moufang loops, VD-loops and universal weak inverse property loops. All these four varieties of Osborn loops are universal [6]. This is what makes non-universal Osborn loops interesting to researchers like Kinyon, Phillips and others [15, 16]. Generally, Osborn loop falls into the class of Bol-Moufang type of loops which play an important role in the theory of quasigroups and in their applications in other branches of Mathematics [7].

2. PRELIMINARY

Theorem 2.1 (Huthnance [9] and Basarab and Belioglo [5]):
Let G be an Osborn loop. $N_\rho(G) = N_\lambda(G) = N_\mu(G) = N(G)$ and $N(G) \trianglelefteq G$.
Lemma 2.1(Huthnance [9]): Every Moufang loop is an Osborn loop.

Lemma 2.2(Huthnance [9]): An Osborn loop that is flexible or which has the LAP or RAP or LIP or RIP or AAIP is a Moufang loop. But an Osborn loop that is commutative or which has the CIP is a commutative Moufang loop.
Remark 2.1: The theorem helps to determine a non-Moufang Osborn loop.Consider also [2, 3]

2.1. Examples of Osborn Loops

Example 2.1:(Kinyon [14]) The smallest order for which proper(non-Moufang and non-CC) Osborn loops with non-trivial nucleus exists is 16. There are two of such loops.

- Each of the two is a G-loop.
- Each contains as a subgroup, the dihedral group(D_4) of order 8.
- For each loop, the center coincides with the nucleus and has order 2. The quotient by the center is a non-associative CC-loop of order 8.
- The second center is $\mathbb{Z}_2 \times \mathbb{Z}$, and the quotient is $\mathbb{Z}_4$.
- One loop satisfies $L_x^4 = R_x^4 = I$, the other does not.

The multiplication tables are presented below in form of acceptable loops as Table 1 and Table 2 .

These two Osborn loops are Smarandache loops(that is, a loop that has at least a non-trivial subgroup). The Smarandache subgroup in each of them is the dihedral group(D_4) of order 8. Detail of this is presented in [11]

Table. 1: The first Osborn loop of order 16 that is a G-loop

$\cdot$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	5	6	8	7	1	2	4	3	13	14	16	15	10	9	11	12
6	6	5	7	8	2	1	3	4	14	13	15	16	9	10	12	11
7	7	8	6	5	3	4	2	1	15	16	14	13	12	11	9	10
8	8	7	5	6	4	3	1	2	16	15	13	14	11	12	10	9
9	9	10	11	12	15	16	13	14	5	6	7	8	3	4	1	2
10	10	9	12	11	16	15	14	13	6	5	8	7	4	3	2	1
11	11	12	9	10	13	14	15	16	8	7	6	5	2	1	4	3
12	12	11	10	9	14	13	16	15	7	8	5	6	1	2	3	4
13	13	14	16	15	12	11	9	10	1	2	4	3	7	8	6	5
14	14	13	15	16	11	12	10	9	2	1	3	4	8	7	5	6
15	15	16	14	13	10	9	11	12	4	3	1	2	6	5	7	8
16	16	15	13	14	9	10	12	11	3	4	2	1	5	6	8	7

Table. 2: The second Osborn loop of order 16 that is a G-loop

$\odot$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	5	6	8	7	1	2	4	3	13	14	16	15	10	9	11	12
6	6	5	7	8	2	1	3	4	14	13	15	16	9	10	12	11
7	7	8	6	5	3	4	2	1	15	16	14	13	12	11	9	10
8	8	7	5	6	4	3	1	2	16	15	13	14	11	12	10	9
9	9	10	11	12	15	16	13	14	7	8	5	6	2	1	4	3
10	10	9	12	11	16	15	14	13	8	7	6	5	1	2	3	4
11	11	12	9	10	13	14	15	16	6	5	8	7	3	4	1	2
12	12	11	10	9	14	13	16	15	5	6	7	8	4	3	2	1
13	13	14	16	15	12	11	9	10	3	4	2	1	6	5	7	8
14	14	13	15	16	11	12	10	9	4	3	1	2	5	6	8	7
15	15	16	14	13	10	9	11	12	2	1	3	4	7	8	6	5
16	16	15	13	14	9	10	12	11	1	2	4	3	8	7	5	6

Table. 3: The Smarandache Subgroup(D_4) of an Osborn loop

$\cdot/\odot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	2	1	8	7	6	5
5	5	6	8	7	1	2	4	3
6	6	5	7	8	2	1	3	4
7	7	8	6	5	3	4	2	1
8	8	7	5	6	4	3	1	2

Table. 4: The first latin sub-square of length 8 from the first and second Osborn loops

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	8	7	1	2	4	3
6	5	7	8	2	1	3	4
7	8	6	5	3	4	2	1
8	7	5	6	4	3	1	2

Table. 5: The second latin sub-square of length 8 from the first and second Osborn loops

9	10	11	12	13	14	15	16
10	9	12	11	14	13	16	15
11	12	9	10	15	16	13	14
12	11	10	9	16	15	14	13
13	14	16	15	10	9	11	12
14	13	15	16	9	10	12	11
15	16	14	13	12	11	9	10
16	15	13	14	11	12	10	9

3. RESULTS

3.1 Representations of Osborn Loops of order 16

Theorem 3.1: Let Π be the right regular representations of an Osborn loop of order 16. Then, the element(permutation) $R(2)$ and other elements (permutations) of odd numbers greater than 2 ($R(3)$,$R(5)$,..,$R(15)$) that are between 1 and 16 determine the loop.

Proof: Consider an Osborn loop of order 16 represented by Π. Suppose $R(2) \in \Pi$ is given and suppose other elements of odd numbers greater than 2 that are between 1 and 16 (R(3),R(5),R(7),R(9), R(11), R(13) and R(15)) are also given. Then the other elements are generated as follows:
$R(2)^2 = R(3)^2 = R(1) = I$

$$R(4) = R(2) \circ R(3)$$

$$R(6) = R(2) \circ R(5)$$
$$R(8) = R(2) \circ R(7)$$
$$R(10) = R(2) \circ R(9)$$
$$R(12) = R(2) \circ R(11)$$
$$R(14) = R(2) \circ R(13)$$
$$R(16) = R(2) \circ R(15)$$

Thus, R(1),R(2),...,R(16) is an Osborn loop of order 16.
Remark 3.1: Thus R(2),R(3),R(5),R(7),R(9), R(11), R(13) and R(15) are the determining permutations of Osborn loops of order 16
Corollary 3.1: Let Π be the representation of an Osborn loop of order 16, and a transposition permutation $R(2) \in \Pi$ such that $R(2)^2 = I$. Then given the determining permutations in Π, others are generated by: $R(2) \circ R(2+i) = R(3+i)\ \forall\ i = 1, 3, 5, ..., 13.$, and $R(2+i)$ determines the structure and order of $R(3+i)$.i.e. $R(3+i)$ retains the structure and order of $R(2+i)$ where $i = 1, 3, 5, .., 13$.
Proof: When $i = 1$, the equation becomes

$$R(2) \circ R(3) = R(4)$$

when $i = 3$, we have

$$R(2) \circ R(5) = R(6)$$

when $i = 5$, we have

$$R(2) \circ R(7) = R(8)$$

continuing in this way up to $i = 13$, we have

$$R(2) \circ R(15) = R(16)$$

The composition follows from theorem 3.1 . Hence, the proof.
Remark 3.2: If the given determining permutations are of even numbers then the equation becomes $R(2) \circ R(2+i) = R(1+i)\ \forall\ i = 0, 2, 4, ..., 14$.
Example 3.1: Given the following:

$$R(2) = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)$$

$$R(3) = (1, 3)(2, 4)(5, 8)(6, 7)(9, 11)(10, 12)(13, 16)(14, 15)$$
$$R(5) = (1, 5)(2, 6)(3, 7)(4, 8)(9, 15, 10, 16)(11, 13, 12, 14)$$
$$R(7) = (1, 7, 2, 8)(3, 5, 4, 6)(9, 13)(10, 14)(11, 15)(12, 16)$$
$$R(9) = (1, 9, 7, 15, 2, 10, 8, 16)(3, 11, 6, 14, 4, 12, 5, 13)$$

$$R(11) = (1, 11, 8, 13, 2, 12, 7, 14)(3, 9, 5, 16, 4, 10, 6, 15)$$
$$R(13) = (1, 13, 6, 9, 2, 14, 5, 10)(3, 15, 7, 12, 4, 16, 8, 11)$$
$$R(15) = (1, 15, 6, 12, 2, 16, 5, 11)(3, 13, 7, 9, 4, 14, 8, 10)$$

determine an Osborn loop of order 16

Solution: Using Corollary 3.1 , we obtained the following permutations

$$R(1) = R(2)^2 = I$$

$$R(4) = (1, 4)(2, 3)(5, 7)(8, 6)(9, 12)(10, 11)(13, 15)(14, 16)$$
$$R(6) = (1, 6)(2, 5)(3, 8)(4, 7)(9, 16, 10, 15)(11, 14, 12, 13)$$
$$R(8) = (1, 8, 2, 7)(3, 6, 4, 5)(9, 14)(10, 13)(11, 16)(12, 15)$$
$$R(10) = (1, 10, 7, 16, 2, 9, 8, 15)(3, 12, 6, 13, 4, 11, 5, 14)$$
$$R(12) = (1, 12, 8, 14, 2, 11, 7, 13)(3, 10, 5, 15, 4, 9, 6, 16)$$
$$R(14) = (1, 14, 6, 10, 2, 13, 5, 9)(3, 16, 7, 11, 4, 15, 8, 12)$$
$$R(16) = (1, 16, 6, 11, 2, 15, 5, 12)(3, 14, 7, 10, 4, 13, 8, 9)$$

Thus, $R(1), ..., R(16)$ is an Osborn loop of order 16.

Remark 3.3: The Osborn loop in the example correspond to Osborn loops by Kinyon as presented in [14, 15]

Corollary 3.2: Let Π be the representation of an Osborn loop of order 16, and $R(2)$ a transposition permutation in Π. Then given the determining permutations in Π, others are generated by $< R(2), R(2 + i) >$ where i is either an even or odd number depending on whether the given determining permutations are of either even or odd number.

Proof: Obviously, as $i = 1, 3, 5, ..., 13$, the determining permutations are given, and the proof follows from theorem 3.1 and corollary 3.1 .

Remark 3.4: For Osborn loops of order n, the formula becomes $\Pi =< R(2), R(2 + i) >$ depending on the available(given) determining permutations, and $< R(2), R(2 + i) >= \{R(2), R(3), R(5), ...R(n - 3)\}$ is the set of generators or determining permutations, where i is an odd number. Or $< R(2), R(2 + i) >= \{R(2), R(4), R(6), ...R(n - 2)\}$ where i is an even number.

Corollary 3.3: If Π is the representations of an Osborn loop of order n, Π has $n/2$ determining permutations.

Proof: Given $R(2)$ in Π, the odd numbers between 1 and n that are greater than 2 will be $(n - 2)/2$ (i.e. n less 1 and 2). Then adding that of $R(2)$ to this number gives $(n - 2)/2 + 1 = n/2$ implies $1/2(n)$ determining permutations.

We need to show by induction that it is true for all values of n.

Suppose $n = 16$, then by theorem above, there are 8 determining permutations, which implies $1/2(16) = 16/2$. So it is true for $n = 16$.

Suppose $n = k$, then we would have $k/2$, implies $1/2(k)$. So, it is true for $n = k$.

Suppose $n = k + 1$, then, we have $(k + 1)/2 = k/2 + 1/2 = 1/2(k + 1)$.

So, it is true for $n = k + 1$.
Inductively, it is true for all values of n. The proof is complete.
Theorem 3.2: Let Π be the representations of an Osborn loop of order 16. Every permutation in Π has no distinct inverse.
Proof: Considering the Osborn loops generated in theorem 3.1 and in example 3.1 , we observe that they have no distinct inverses.
lemma 3.1: Let Π be the representations of an Osborn loop of order 16. The order of every element of the representations Π divides the order of the loop.
Proof: The representations of Osborn loops in Table 1 and Table 2 show that the order of elements of the first example of Osborn loop of order 16 are 2 and 4 while the order of elements of the second example are 2 and 8. These are divisors of 16. The proof follows.
Corollary 3.4: The representations of a finite Osborn loop do not generate a multiplicative group.
Proof: Since the representations of an Osborn have no inverses. Then, they do not form a group. The proof follows.
Remark 3.5: Corollary 3.4 is confirmed by LOOPs Package in GAP [8]

4. SUMMARY

This work divides the search space of an Osborn loop by 2. One only need to generate the determining permutations by any means and using the equation in the corollary above one can get the entire loop.

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